

**ON THE STRUCTURE OF
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IN A
PLASMA TRANSVERSE
TO A MAGNETIC FIELD**

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On the Structure of a Collision Free Wave in a

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Abstract. In this paper we have considered the problem of the structure of a strong collision free wave in a plasma perpendicular to its magnetic field. It is shown that conditions on the two sides of the wave are the same and are separated by a solitary pulse provided the kinetic energy of the motion at one end of the wave ($x = -\infty$, say) is less than the sum of magnetic and thermal energy of the plasma. If the kinetic energy is more than the sum of magnetic and thermal energy, no wave exists and the constant solution at $x = -\infty$ is the only solution of the equations describing the wave form.

AUTHOR:

I. Introduction.

The subject of the structure of a strong collision free wave, in a plasma perpendicular to the magnetic field has been treated by Adlam and Alan¹, Burgers² and Hain, Lüst and Schlüter³. Hain, Lüst and Schlüter in particular have obtained solutions of the wave structure by solving the non-linear equations on a computer.

In this paper we have investigated mathematical properties of the equations which help us to guess the correct solution out of the many solutions obtained on a computer. As an illustrative example we shall consider a plasma with isotropic pressure. The generalization to the anisotropic case is quite straight forward. Burgers has made an extensive study of the wave structure when the plasma pressure is anisotropic and he arrives at the same general result that there is no change of state on the two sides of the wave i.e., that no shock wave exists in a collisionless plasma perpendicular to the magnetic field.

The result of our paper is that a solitary wave only exists if the kinetic energy of motion, which produces the wave, is less than the sum of magnetic and thermal energy of the plasma; that no stationary wave exists if the kinetic energy of motion is more than the sum of magnetic and thermal energy of the plasma. In the latter case the constant solution at one end of the wave is the only solution of the differential equation (section 4).

2. Equations Describing the Wave Form.

Consider a stream of plasma at $x = -\infty$ with a constant velocity u_0 along x-axis and a constant magnetic field B_1 along z-axis. The electrons and ions of the plasma will drift in the y-direction because of the magnetic field.

Our object is to obtain conditions at $x = +\infty$ and the form of the wave or waves which start from the given solution at $x = -\infty$. As a simplification we assume that the electrons and ions satisfy Maxwellian distribution function with variable number density, velocity components and the temperature. Therefore, if f_e , f_i denote the electron and ion distribution functions

$$f_e(x, v) = n_e(x) \left(\frac{m}{2\pi k T_e(x)} \right)^{3/2} \exp \left\{ -\frac{m}{2k T_e(x)} \left[(u - u_e(x))^2 + (v - v_e(x))^2 + w^2 \right] \right\} \quad (1)$$

$$f_i(x, v) = n_i(x) \left(\frac{M}{2\pi k T_i(x)} \right)^{3/2} \cdot \exp \left\{ -\frac{M}{2k T_i(x)} \left[(u - u_i(x))^2 + (v - v_i(x))^2 + w^2 \right] \right\} \quad (2)$$

In cartesian coordinates, assuming steady state, the Boltzmann equation can be written as

$$m u_e \frac{\partial f_e}{\partial x} - e E_x \frac{\partial f_e}{\partial u} - e E_y \frac{\partial f_e}{\partial v} - \frac{e B}{c} \left(v \frac{\partial f_e}{\partial u} - u \frac{\partial f_e}{\partial v} \right) = 0 \quad (3)$$

for electrons, and

$$M u \frac{\partial f_i}{\partial x} + e E_x \frac{\partial f_i}{\partial u} + e E_y \frac{\partial f_i}{\partial v} + \frac{e B}{c} \left(v \frac{\partial f_i}{\partial u} - u \frac{\partial f_i}{\partial v} \right) = 0 \quad (4)$$

for the ions. Multiplying by $1, u_e, v_e, u_e^2$ and $u_e^2 + v_e^2 + w_e^2$ and generating moments of equation (3) we obtain

$$\frac{\partial}{\partial x} [n_e(x) u_e(x)] = 0 \quad (5)$$

$$m n_e u_e \frac{du_e}{dx} + \frac{d}{dx} (n_e k T_e) + n_e e E_x + \frac{1}{c} n_e e v_e B = 0 \quad (6)$$

$$m n_e u_e \frac{dv_e}{dx} + n_e e E_y - \frac{1}{c} n_e e B u_e = 0 \quad (7)$$

$$\frac{d}{dx} \left(u_e^2 + \frac{3 k T_e}{m} \right) + \frac{2 e E_x}{m} + \frac{2 e B}{m c} v_e = 0 \quad (8)$$

$$\frac{d}{dx} u_e \left[m n_e u_e^2 + m n_e v_e^2 + 5 n_e k T_e \right] + 2 e n_e u_e E_x - 2 e n_e v_e E_y = 0 \quad (9)$$

and similar equations for the ions. Let us call the similar ion equations as (5)' - (9)'. Maxwell's equations yield

$$E_y = \text{Constant} \quad (10)$$

$$\frac{dE_x}{dx} = 4\pi e(n_i - n_e) \quad (11)$$

$$\frac{dB}{dx} = -\frac{4\pi e}{c}(n_i v_i - n_e v_e) \quad (12)$$

To further simplify the problem, we will make the assumption that the space charge can be neglected. The characteristic length of space charge oscillations is of the order of the Debye length⁴ $R_D = \left(\frac{kT}{4\pi n e^2}\right)^{1/2}$ while, as we shall see in this paper, that of magneto-hydrodynamic oscillations is of the order of $c/\omega_p = c \sqrt{\frac{m}{4\pi N e^2}}$.

Thus assuming that the electron thermal velocity is well below the velocity of light, the space charge oscillations are of much lower scale than the oscillations we are interested in. With the assumption $E_x \approx 0$ and $n_e = n_i = n$ (to a good approximation we then have $u_e = u_i = u$, say), equation (6), (6)' and (8), (8)' yield, respectively

$$\frac{d}{dx} \left[M n u_i^2 + n_i k T_i + n k T_e \right] = -\frac{1}{8\pi} \frac{dB^2}{dx} \quad (13)$$

$$\frac{d}{dx} \left[M u_i^2 + 3 k T_i + 3 k T_e \right] = -\frac{1}{2\pi} \frac{B}{n} \frac{dB}{dx} \quad (14)$$

Equation (5) now gives

$$n u = n_1 u_1 \quad (15)$$

where $\lim_{x \rightarrow -\infty} n(x) = n_1$ and $\lim_{x \rightarrow -\infty} u(x) = u_1$.

Equation (13) can be integrated and yields

$$M n u^2 + n k T_i + n k T_e + \frac{B^2}{8\pi} = M n_1 u_1^2 + 2 n k T_i + \frac{B_1^2}{8\pi} \quad (16)$$

where $\lim_{x \rightarrow -\infty} B(x) = B_1$. Eliminating $T_i + T_e$ from equations (16)

and (14) we obtain

$$u \frac{d}{dx} \left[u \frac{dB}{dx} \right] + \frac{4\pi n_1 e^2}{m c^2} u_1 u B - \frac{4\pi n_1 e^2}{m c^2} u_1^2 B_1 = 0 \quad (17)$$

Dividing by $\sqrt{8\pi M n_1 u_1^2}$ and using the transformations

$$\frac{u}{u_1} = U, \quad \frac{B}{\sqrt{8\pi n_1 M u_1^2}} = H, \quad \frac{k T_{i,e}}{M u_1^2} = \theta_{i,e}, \quad \beta_1^2 = \frac{B_1^2}{8\pi n_1 M u_1^2} \quad (18)$$

we finally obtain the equations

$$H^2 = \alpha - U - \frac{2\theta_1}{U^3} \quad (19)$$

and

$$U \frac{d}{dy} \left(U \frac{dH}{dy} \right) + U H - \beta_1 = 0 \quad (20)$$

where

$$\alpha = 1 + 2\theta_1 + \beta_1^2 \quad (21)$$

Equations (19), (20) satisfy the boundary conditions $H = \beta_1, U = 1$ at $y = -\infty$

3. A Mathematical Discussion of Equations (19), (20)

The pair $(U, H) = (1, \beta_1)$ is a constant solution of these equations. We shall study other constant solutions represented by (19), (20), but we first examine the stability of the solutions of (19), (20) in the neighborhood of $(1, \beta_1)$ and determine the nature of this critical point.

Let $U = 1 + U_1, H = \beta_1 + H_1$. We then obtain from (19)

$$H_1 = U_1 F(H_1, U_1) \quad (22)$$

where

$$F(H_1, U_1) = \frac{2\theta_1(3 + 3U_1 + U_1^2) - (1 + U_1^3)}{(2\beta_1 + H_1)(1 + U_1)^3} \quad (23)$$

From (22) and (23) we obtain

$$H_1 = b_1 U_1 + b_2 U_1^2 + b_3 U_1^3 + \dots \quad (24)$$

where

$$b_1 = F(0,0) = \frac{6\theta_1 - 1}{2\beta_1} \quad (25)$$

$$\begin{aligned} b_2 &= F(0,0) \frac{\partial F}{\partial H_1}(0,0) + \frac{\partial F}{\partial U_1}(0,0) \\ &= -\frac{(6\theta_1 - 1)^2}{8\beta_1^3} - \frac{6\theta_1}{\beta_1} \end{aligned} \quad (26)$$

etc.

Equations (20), (24) yield,

$$\frac{d^2 U_1}{dy^2} + \gamma_1(U_1) \left(\frac{dU_1}{dy}\right)^2 + K U_1 + \gamma_2(U_1) = 0 \quad (27)$$

where γ_1, γ_2 are analytic in U_1 near $U_1 = 0$, $\gamma_2(U_1) = O(U_1^2)$ and

$$K = 1 + \frac{\beta}{b_1} = \frac{1 - 2\beta_1^2 - 6\theta_1}{1 - 6\theta_1} \quad (28)$$

To determine the nature of the point $U = 1$, we linearize equations (23), (27). The linearized form of equation (27) is

$$\frac{d^2 U_1}{dy^2} + K U_1 = 0 \quad (29)$$

In all the discussion that follows we shall assume the flow velocity to be much higher than the thermal velocity of particles constituting the plasma, so that $6\theta_1 < 1$ or $1 - 6\theta_1 > 0$. The sign

of k^1 is therefore determined by the expression $1 - 2\beta_1^2 - 6\theta_1$.
 In case I when $1 - 2\beta_1^2 - 6\theta_1 > 0$, k^1 is positive and therefore $U = 1$ is a center in both (27) and (29) according to a known result of Lyapounov, see Malkin⁵, pp 123-4, (as no odd powers of dU_1/dy occur).
 In case II when $1 - 2\beta_1^2 - 6\theta_1 < 0$, k^1 is negative; the point $U = 1$ is a saddle point in both equations (27) and (29).

Let us now consider all the constant solutions represented by the equations (19), (20). According to equation (20) constant solutions satisfy the equation $HU = \beta_1$. This condition, together with equation (19) then gives the equation

$$f(U) \equiv U^4 - U^3(1 + 2\theta_1 + \beta_1^2) + \beta_1^2 U + 2\theta_1 = 0 \quad (30)$$

$f(U)$ may also be written as

$$\begin{aligned} f(U) &= (U-1)[U^3 - (\beta_1^2 + 2\theta_1)(U^2 + U) - 2\theta_1] \\ &= (U-1)h(U) \end{aligned} \quad (31)$$

From Descarte's rule of signs $h(U)$ cannot have more than one positive root. Also it is easy to see that it has a positive root. Let this positive root be denoted by U_0 . Since $f(0) < 0$, U_0 will be less than or greater than 1 when $f'(1) > 0$, $f'(1) < 0$ respectively. We therefore again have two cases:

$$(I) \quad f'(1) = 1 - 2\beta_1^2 - 6\theta_1 > 0$$

and

$$(II) \quad 1 - 2\beta_1^2 - 6\theta_1 < 0.$$

As we require that $6\theta_1 < 1$, Case I may occur. Figure 1 shows a rough sketch of $f(U)$ versus U for cases I and II above.

To find the nature of the other critical point $U = U_0$, we write equation (20) as

$$\frac{d^2U}{dy^2} + \left(\frac{dU}{dy}\right)^2 \left[\frac{Ug''(U) + g'(U)}{Ug'(U)} \right] + \frac{Ug(U) - \beta_1}{U^2g'(U)} = 0 \quad (32)$$

where

$$H \equiv g(U) = \left[a - U - \frac{2\theta_1}{U^3} \right]^{1/2} \quad (33)$$

$g'(U)$ has a zero at $U^* = (6\theta_1)^{1/4}$. It is interesting to note that this singularity of the differential equation can be quite close to 1. For example, for $\theta_1 = .1$, $(6\theta_1)^{1/4} \approx .880117$. Higher values of θ_1 can bring this singularity still closer to $U = 1$.

When equation (32) is linearized about the zero U_0 of $f(U)$, we obtain the equation

$$\ddot{U}_1 + L(U_0) U_1 = 0 \quad (34)$$

where $U = U_0 + U_1$ and $U L(U) = 1 + \frac{2\beta_1^2}{U^2(6\theta_1 - U^4)}$. The nature of the point $U_0 = 1$ has already been discussed. Now let us assume in addition to $6\theta_1 < 1$, that $U_0^4 > 6\theta_1$. We wish to establish for equation (32) that in case I when the point $U = 1$ is a center the other equilibrium point $U_0 < 1$ is a saddle point and conversely, in case II when $U = 1$ is a saddle point, the other equilibrium point $U_0 > 1$ is a center. By arguments similar to those given above for equation (27) and (29), this is equivalent to showing that if $h(U_0) = 0$, h given in (31), then $U_0 < 1$ implies $L(U_0) < 0$ and $U_0 > 1$ implies $L(U_0) > 0$.

It can be shown that

$$U g(U) - \beta_1 = \frac{-f(U)}{U [\beta_1 + U g(U)]} \quad (35)$$

Now, consider case I. $f(U)$ has a zero at $U = U_0 (< 1)$ and $U = 1$. In the U, \ddot{U} plane construct a closed loop C enclosing U_0 and 1 and lying so close to the U -axis that the sign of \ddot{U} is determined by the last term on the left in (32). (see Figure 2). Then using (35) and upon considering the changes in the sign of $f(U)$, it follows that the index of C with respect to (32) is 0^6 . As the index of a center ($U = 1$) is 1, this implies that the index of U_0 is -1. If $L(U_0) \neq 0$, i.e., if U_0 is an elementary critical point, then U_0 of necessity must be a saddle point and $L(U_0) < 0$ will hold. A similar arrangement for $U_0 > 1$ in case II shows that the index of U_0 is 1 and therefore $L(U_0) > 0$ holds (provided of course $L(U_0) \neq 0$).

Thus the problem is reduced to showing that the two polynomials $h(U) = U^3 - bU^2 - bU - 2\theta$, and $M(U) = U^6 - 6\theta_1 U^2 + 4\theta_1 - 2b$ have no common positive zeros in either case I or case II. Here we have used the definition $b = 2\theta_1 + \beta_1^2$ in deriving the form of $M(U)$ which is the numerator of $UL(U)$.

This has not been shown in general, although for particular choices of β_1, θ_1 , it is relatively straightforward. The critical case, however, is when $b + \theta_1 = \frac{1}{2}$ or $2\beta_1^2 + 6\theta_1 = 1$, for then the two roots U_0 and 1 become equal and, of course, in this case $M(1) = 0$.

Let us suppose that $b + \theta_1$ is close to $\frac{1}{2}$, i.e., U_0 is close to 1 so that $(U_0 - 1)^2, (U_0 - 1)^3$ are negligible in comparison with $U_0 - 1$. Now let $R = U^2$ and expand h and M about $U = 1$ and $R = 1$. This gives

$$h(U) = 1 - 2(b + \theta_1) + 3(1 - b)(U - 1) + (3 - b)(U - 1)^2 + (U - 1)^3 \quad (36)$$

and

$$M(R) = 1 - 2(b + \theta_1) + 3(1 - 2\theta_1)(R - 1) + 3(R - 1)^2 + (R - 1)^3 \quad (37)$$

The zeros U_0, R_0 ($=\bar{U}^2 - 1$) of h and M may be approximated by

$$U_0 = 1 + \frac{2(b + \theta_1) - 1}{3(1 - b)} \quad (38)$$

$$R_0 = 1 + \frac{2(b + \theta_1) - 1}{3(1 - 2\theta_1)}$$

Now suppose we are in case I, that is $b + \theta_1 < \frac{1}{2}$. Then as $b > 2\theta_1$, we have $U_0 - 1 < R_0 - 1 = \bar{U}^2 - 1 < 0$; where $M(\bar{U}) = 0$, $\bar{U} > 0$. This implies

$$1 - U_0 > (1 - \bar{U})(1 + \bar{U}) > (1 - \bar{U})$$

Therefore

$$U_0 < \bar{U} < 1$$

In case II, $b + \theta_1 > \frac{1}{2}$, it follows similarly from $U_0 - 1 > \bar{U}^2 - 1$, that $1 < \bar{U} < U_0$. Thus the positive roots of M and R , which coincide for $b + \theta_1 = \frac{1}{2}$, do separate as required, for $b + \theta_1$ near $\frac{1}{2}$. Although a proof for the general case is lacking, the following calculations may be of some interest.

Taking $\theta_1 = .1$, $b = .3$, we find $(6\theta_1)^{1/4} = .880117$, $U_0 \cong .88978$ and $M(U_0) = -.1788 < 0$. With $\theta_1 = .1$, $b = .5$, we find $U_0 \cong 1.1116$ and $M(U_0) \cong .54567 > 0$. Thus we observe that at $b = .3$, U_0 is very close to $U^* = (6\theta_1)^{1/4}$ and so b could not be decreased further and still have $U_0^4 > 6\theta_1$.

Thus we have seen that in case II, that is $U_0 > 1$, if $M(U_0) > 0$, then U_0 is a center. In this case the phase plane portrait of the solution is shown in figure 3. In the other case, in which U_0 is a

saddle point and 1 a center, the phase plane picture is the same as figure 3 with the role of 1, U_0 reversed. As is clear from our analysis we have tried to determine the structure of the solutions of equations (19), (20) from the properties of the differential equations (27) or (32). In the next section, we shall verify our predictions by obtaining some solutions on a computer.

4. Computer Calculations

Some of the results given in the previous section were verified by computing the solution of equations (19), (20) on an IBM 7094. The way to do this is to choose values of U , H (different from 1, β_1) lying on the eigen solution by using the linear equation (29). All calculations were started close to $U = 1$, $H = \beta_1$. Since the solution depends critically on the position of the point of start in the phase plane we ran several cases using values slightly different from those obtained from the linear equation (29). For example, for $\beta_1 = .81$, $\theta_1 = .05$, we obtained the solution using (a) $U=1.0001$, $\frac{dU}{dy} = .00001$ and (b) $U=1.0001$, $\frac{dU}{dy} = .001$ while from the linear equation we obtain a value of $\frac{dU}{dy}$ which lies between .00001, .001. Figures 4a and 4b give the solution for the cases (a), (b) above. In the case (b) the solution becomes unstable because we happen to use a value of $\frac{dU}{dy}$ which lies outside the oval of figure 3. In the case (a), we obtain a solution which almost returns back to its initial value and repeats itself. (The repetitions are evidently the cause of the computational error, which if decreased will show that the solution is a solitary pulse. Since only two branches of the curve approach

$U = 1$, $U = 0$ the solitary pulse is the required solution of the wave form.)

We now turn to a calculation of case I considered in section 3. We used $\beta_1 = .1$, $\theta_1 = .1$. Taking a point away from $U = 1$, $\frac{dU}{dy} = 0$, we obtained a solution consisting of oscillations having $U = 1$ $\frac{dU}{dy} = 0$ for its center in accord with the fact that $U = 1$, $\frac{dU}{dy} = 0$ is a center. We see therefore that in case (I) even the solitary wave does not exist.

We have not made any computations about the point $U = U_0$ as this point is of no interest to us.

This corresponds to a solution lying just within the oval pictured in figure 3. A more accurate choice of initial conditions would show that the boundary of this oval corresponds to a solitary pulse, that is, the solution forming the boundary of the oval leaves $U = 1$, $U = 0$ at $y = -\infty$, and returns to this point at $y = +\infty$.

References:

1. J. H. Adlam and J. E. Allen, Phil. Mag. 3, 448 (1958)
2. J. M. Burgers, Rev. Mod. Phys. 32, 868 (1960)
3. K. Hain, R. Lust and A. Schluter, Rev. Mod. Phys. 32
967 (1960)
4. O. W. Greenberg and Y. M. Treve, Phys. Fluids 5, 769 (1960)
5. I. G. Malkin, "Theory of Stability of Motion:" AEC-tr-3352
(Russian Edition 1952)
6. V. V. Nemytskii and V. V. Stepanov, "Quantitative Theory
of Differential Equations", Princeton University Press,
1960 (pp 125-130)

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Figure Captions

- Fig. 1. $f(U)$ vs U for Cases I, II of text.
- Fig. 2. The closed curve C enclosing the points $U = U_0$ and $U = 1$ and lying close to the U axis.
- Fig. 3. Approximate phase plane picture of the solution for the case II when $U = 1$ is a saddle point and $U = U_0$ is a center.
- Fig. 4a. Computed solution of equations (19), (20) for $\beta_1 = .81$, $\theta_1 = .05$, using perturbed values $U = 1.0001$, $\frac{dU}{dy} = .00001$.
- Fig. 4b. Computed solution of equations (19), (20) for $\beta_1 = .81$, $\theta_1 = .05$, using perturbed values $U = 1.0001$, $\frac{dU}{dy} = .001$.

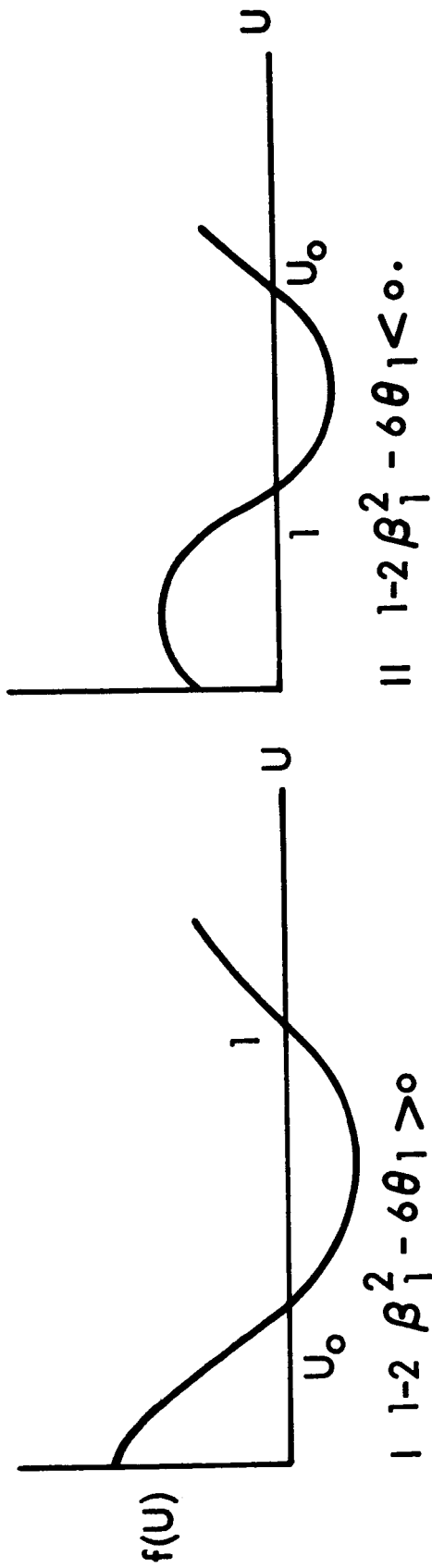


Fig. 1. $f(U)$ vs U for case I, II of text

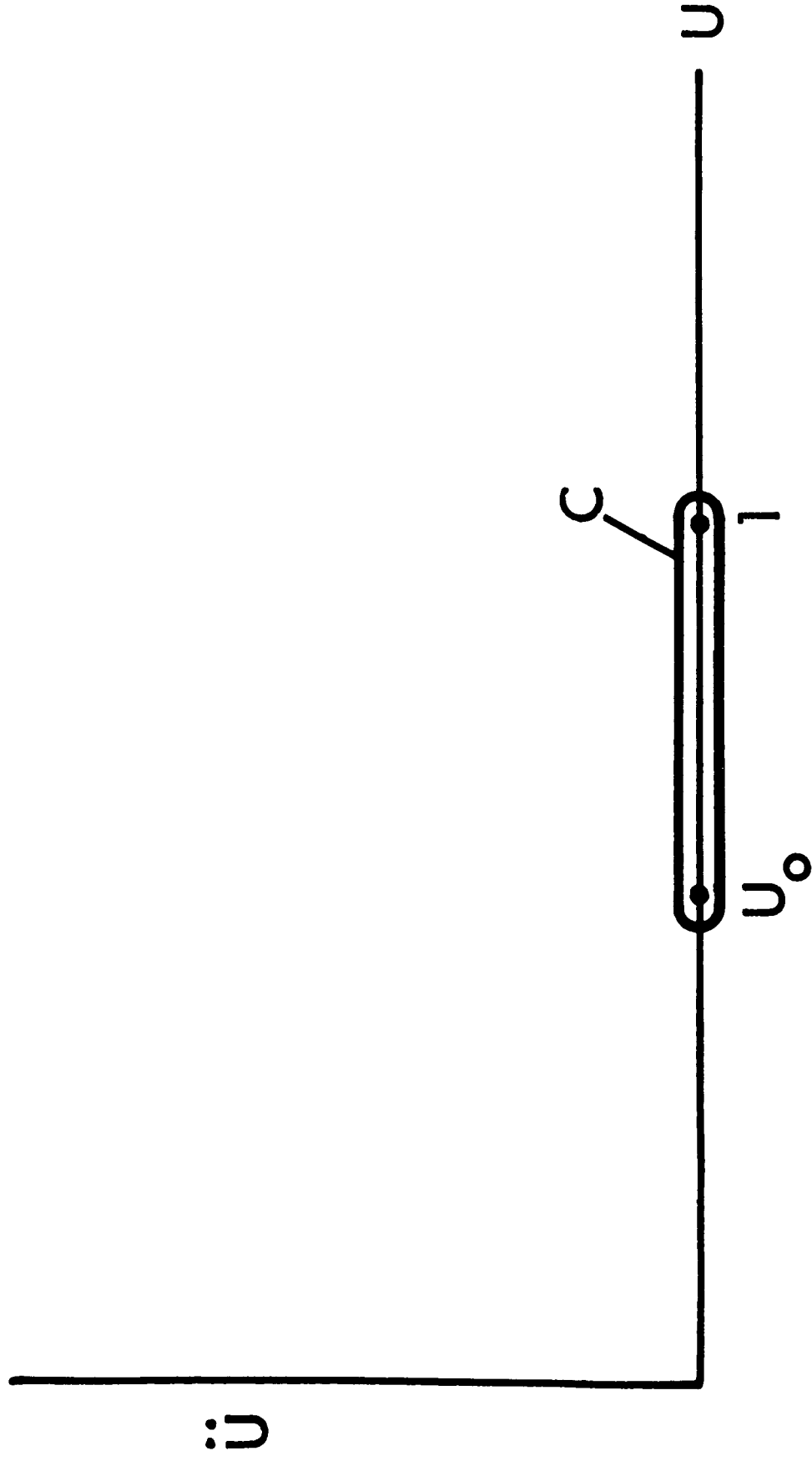


Fig. 2. The closed curve C enclosing the points $U = U_o$ and $U = 1$ and lying close to the U axis.

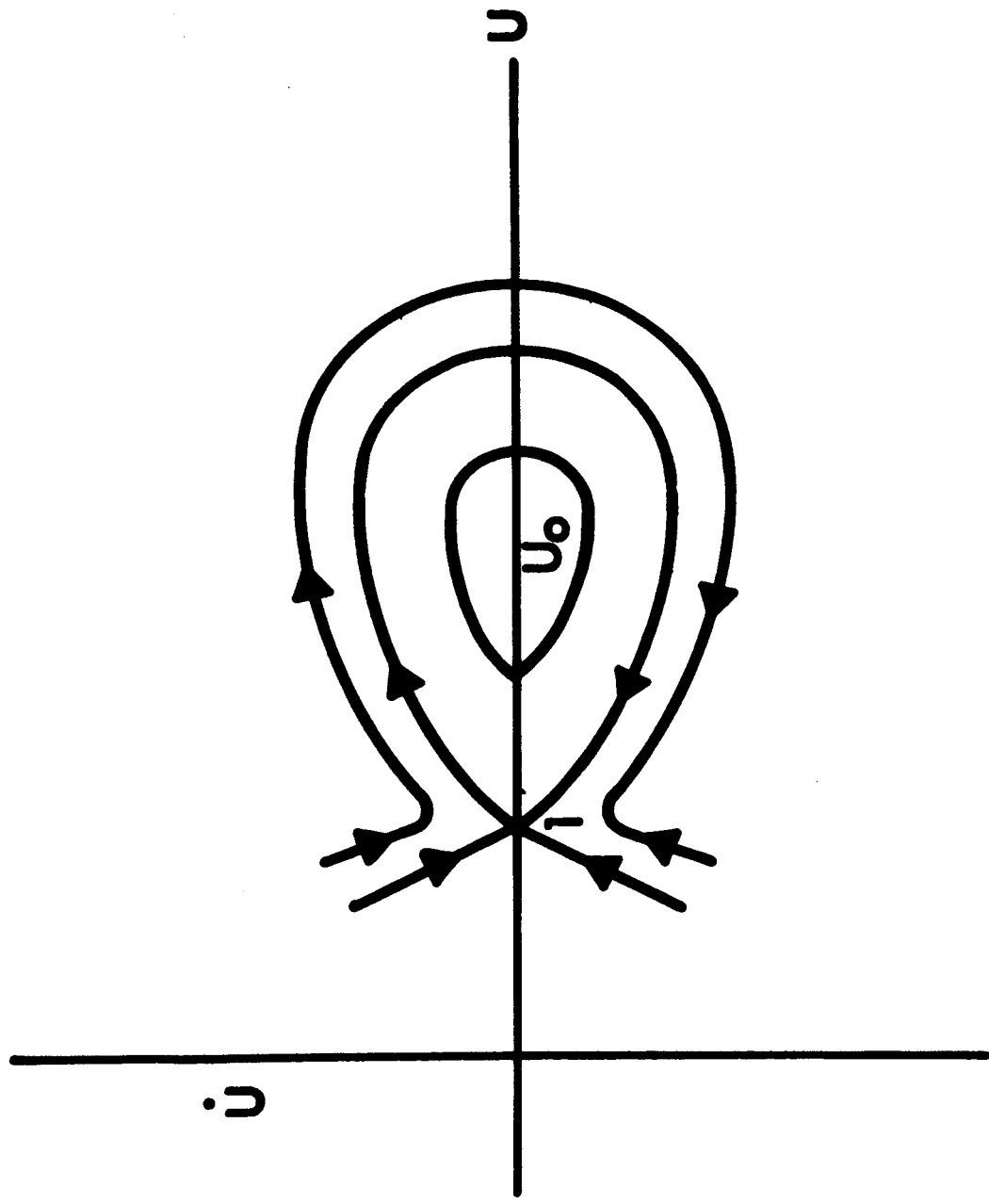


Fig. 3. Approximate Phase plane picture of the solution for the case II when $U = 1$ is a saddle point and $U = U_0$ is a centre.

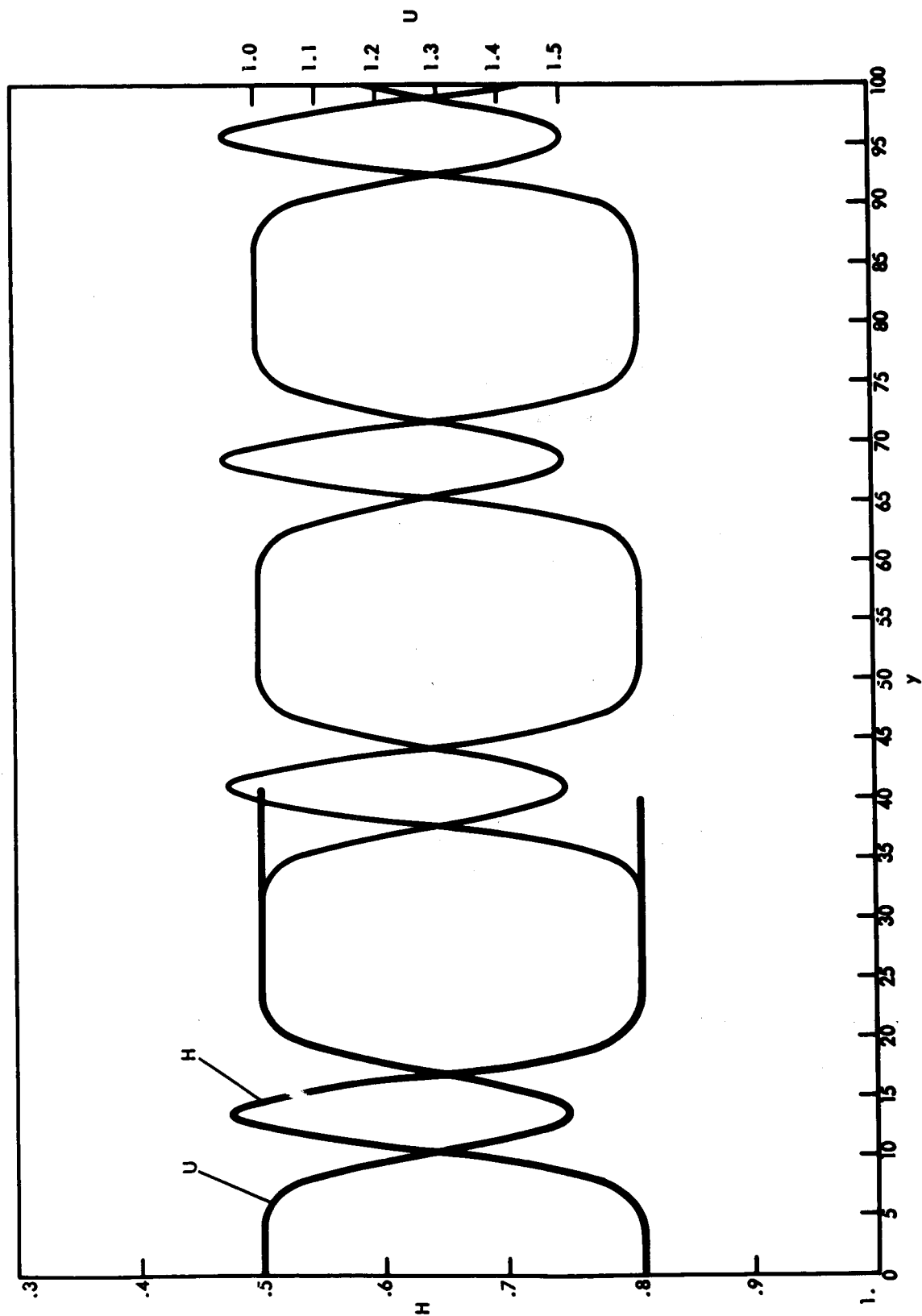


Fig. 4a

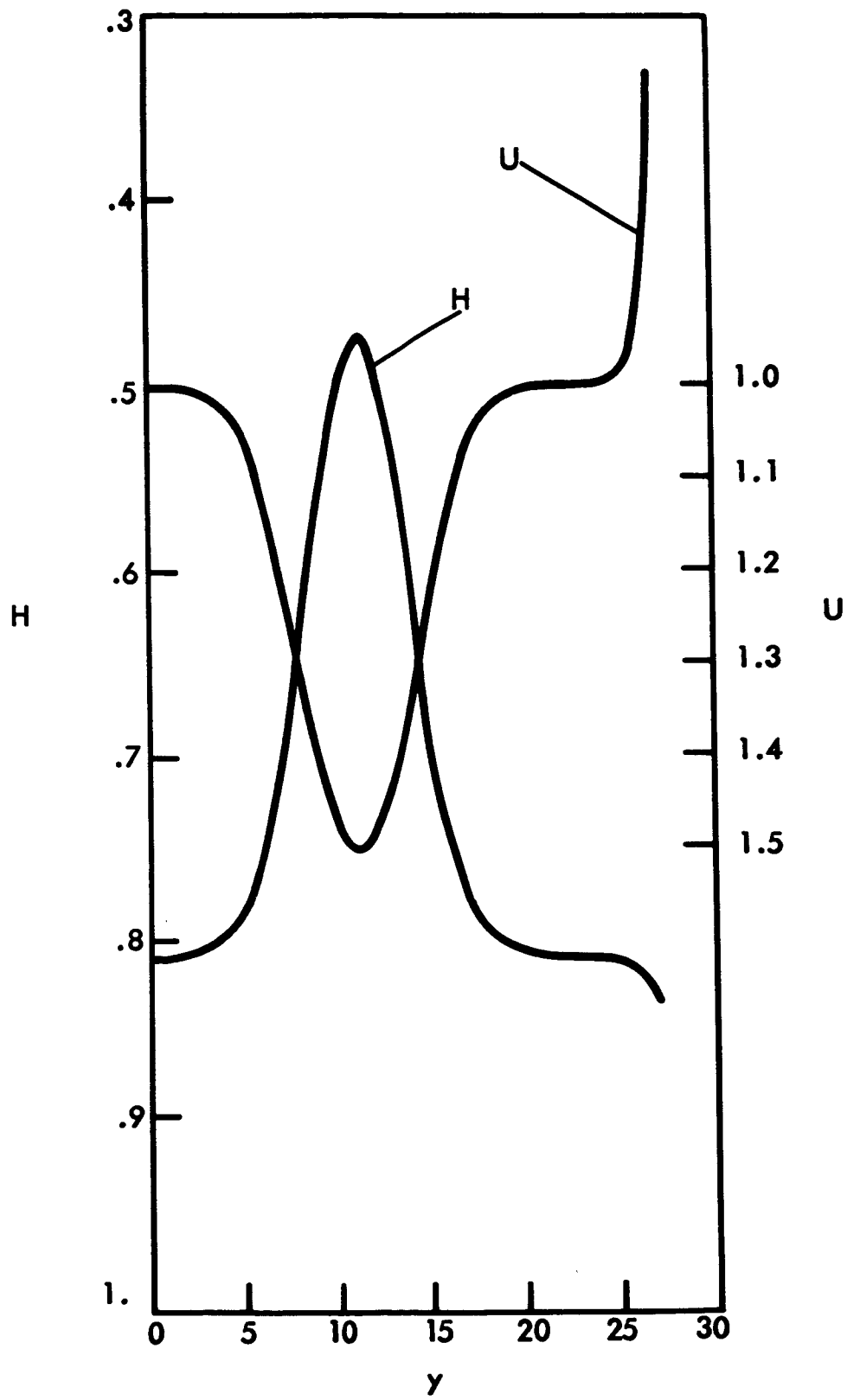


Fig. 4b